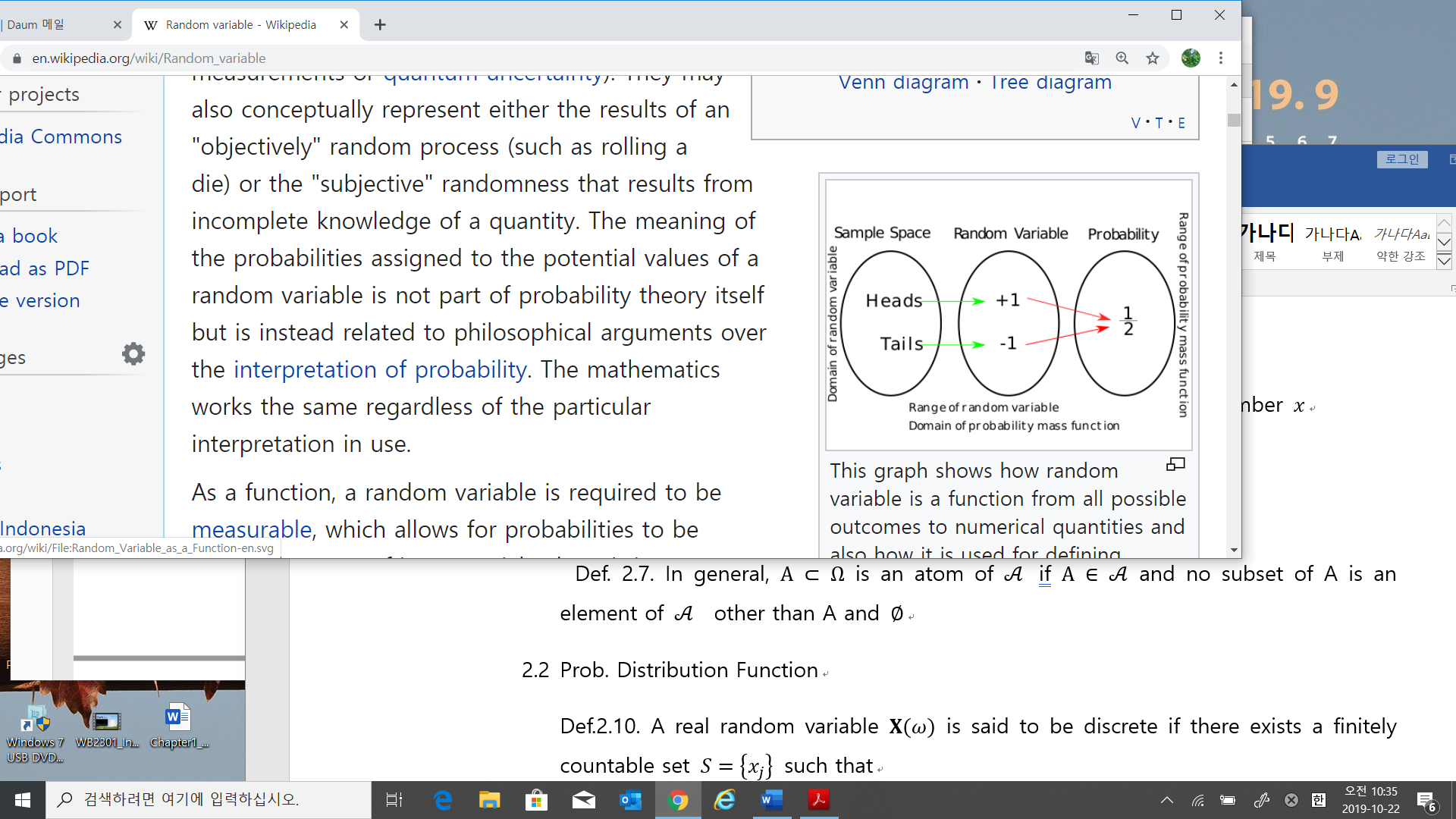
1. Random Variables and Stochastic Process
   1. Random Variables

Def. 2.1. Given a probability space,, a random variable is a real-(vector-) valued point function which carries a sample point, , into a point

in such a way that every sets, , of the form

is an element of the

* In the textbook, mis spelled. As
* Random variable
* is a function such that associates to a real number
* Wiki: <https://en.wikipedia.org/wiki/Random_variable>



Ex.2.4. The Experiment: two coins flips

The sample space

The Event: at least one Head:

A candidate random variable

We may call this as the Indicator random variable

Now the event generated by may be defined (as you like)

Probability graph in

0 1

3/4

1/4

%%% Kim’s comment:

What is in this case? Well

And

Hence we may calculate for any %%%%

Def. 2.7. In general, is an atom of if and no subset of A is an element of other than A and .

Hence are atoms of

%%% Kim’s comment :

See the notation , %%%

* 1. Prob. Distribution Function
* **Probability Distribution function**

Def.2.10. A real random variable is said to be discrete if there exists a finitely countable set such that

* 1. Prob. Density Function

Suppose such that

Then is **the probability density function**,

Proposition 2.12.

%%% Kim’s comment : in the limit notation

%%%

%%% kim’s comment on

One of the special function in mathematics is function. The definition of is

In the shift form

What is the value of ? It may be called as an impulse function. As you see

Now for any constant , which is as large as possible

So to satisfy (1), the magnitude of , which is not defined at the real number.So in fact the delta function is not a function. We may remember in the system theory, the Laplace transform of the delta function, i.e.,

%%%%

* Common (Probability) Distribution Functions for Random Variables

1. The uniform distribution function
2. The exponential distribution function
3. The Gaussian probability distribution(the normal distribution function)

A Gaussian random vector , the density is

where the mean vector,

the covariance matrix,

the determinant of

* 1. Probabilistic Concepts Applied to Random Variables
* Joint Probability Distribution
* Marginal Probability
* Joint Probability Density
* Marginal Probability Density
* The marginal probability distribution
* Ex.(Kim)

Given find the marginal

* Kim’s Example

1. Is it a CDF ?



1. Find
2. Is independent? No since

* Kim Example 2



Is this PDF independent? Yes…. Prove it.

Def 2.16. Two random variables and are called **independent** if any event of the form id independent of any event of the form where are sets in

* Fact
* The joint probability distribution
* The joint probability density function
* Marginal Probability Distribution

Let the joint PDF is

The marginal PDF of

can be calculated as

* 1. Functions of a Random Variable **-skip**

has the density function

Where stands for the absolute value of the determinant of the matrix

* 1. Expectations and Moments of a Random Variable

Def.

* The mean
* The sample mean

%% The same mean is a Random Variable! It is an estimator of the mean of a random variable . If is **independent identical distributed (iid)** random variable,i.e.,

Then the mean of the sample mean is

* Kim’s Comment

What is the difference between a) and b)? In order to use (a) , it is needed know the probability density function, whereas in (b), not needed.

Examp. 2.19. is uniformly distributed from 0 to 1,i.e.,

Then

Examp. 2.22 the expectation of the value of one roll of one die?

Properties

1. The operator of expectation is linear
2. The square mean / second moment
3. The higher order moment
4. The variance
5. The standard deviation
6. The sample variance

This is a random variable. And the **unbiased** estimator of

* Kim’s comment

What is the estimator? Let be a RV. I want to find a constant “C” as RV in some sense.

We may call C as a estimator of the RV . So there may be many estimator as you like.

We may classify the estimator as

1. Unbiased estimator / biased estimator

, then C is the unbiased estimator

1. The minimum variance estimator /the least square error estimator
2. The mean of is the minimum variance estimator / the least square error estimator.

Proof:

* , which minimizes the (c).

Examp. 2.24 The uniform distributed random variable

The Variance is

* 1. Characteristic Functions **-skip**

Lemma 2.27

Prop.2.28 If is a Gaussian random vector with mean, m, and covariance matrix P, then its characteristic function is

* Kim

Def : Two R.V. are **uncorrelated** if

Def: Two Gaussian R.Vectorsare uncorrelated if is a diagonal matrix

**Prop. 2.29. Uncorrelated Gaussian random variables are independent**

Theorem 2.30. If is a Gaussian random vector with mean , and covariance, , and if , where is a Gaussian random vector with zero mean and covariance, , then is a Gaussian random vector with mean, , and covariance, .

Theorem 2.30

**A R.V , another R.V. and they are independent**. Find mean and covariance of

* Characteristic function is difficult to remember. In the text book, using the characteristic method. In this case we may apply basic theory.

Sol: Let’s apply the basic definition.

Hence

* Theorem 2.30 is important. But **the assumption in the theorem is insufficient** as

**are independent.**

* In general, independency implies the uncorrelated, not vice versa
* However, in Gaussian Does satisfy the opposite direction.
* The covariance of a uncorrelated (so independent) Gaussian is a diagonal matrix,
* Linear matrix theory: similar transform

We know for any semi-positive symmetric matrix , there is a similar transform matrix such that

Hence the covariance for any gaussian Random vectors (correlated), there exits a such that

* Any Gaussian Random vectors, we can find a transformed Random Vectors which is uncorrelated (independent).
* Independency is important to calculate the probability. You know the Gaussian probability table, but it is a scalar. So it you want to calculate the joint probability which may be correlated, first find a similar transform matrix to generate a diagonal covariance matrix. Then you may calculate the joint probability as a separate probability.
* **The central limit theorem**

Theorem 2.31. Let be i.i.d. random variables with finite mean and variance,

and denote their sum as . Then the distribution of the normalized sum

is a Gaussian distribution with mean 0 and variance 1 in the limit as

* Proof : textbook P.52
* Remarks:

1. See, the condition, that means   
   the mean and the variance is constant, but the experiment is many time processing. For example,
2. A die, which is fair or not, you roll the same die many times. Then the mean of the sum () is a Gaussian if .
3. Some RV has no mean, then it will not be applicable.
   1. Conditional Expectations and Conditional Probabilities

* The conditional expectation
* Remarks
* is a constant, means it is not random variable.
* if is a constant, then is a constant
* if is a RV, then is a **Random Variable** of y
* **Iterated expectation** **(See the proof at p.57 and remember)**
* Kim’s comment

Even if we do not know .

* Kim Examp.

. , 🡪

Lemma 2.34.

* 1. Stochastic Process

Def. 2.36. A stochastic process is a family of random variables, , indexed by a real parameter and defined on a common probability space .

Ex. 2.37

Def. 2.38.

1. A stochastic process is said to be continuous in probability at t if

for all

1. A stochastic process is said to be separable if there exists a countable, dense set such that for any closed set

differ by a set such that

Theorem 2.40. The rational numbers in provide a separating set S.

Def. 2.42. Let X be a random process defined on the time interval, T. Let

be a partition of the time interval, T. If the increments, are mutually independent for any partition of T, then X is said to be a process with **independent increments**.

Def. 2.43 We say that a random process, X, is a Gaussian process if for every finite collection, the corresponding density function,

is a Gaussian density function.

Def. 2.44 We say that a random process X is a Gaussian process if every finite linear combination of the form

is a Gaussian random variable

Def 2.45. A random process, where T is a subset of the real line, is said to be a **Markov process** if for any increasing collection

or, equivalently

* 1. Gauss-Markov Processes – **The fundamental**

1. Dynamics

* State , is a known matrix, is a Gaussian Random sequence.

1. Given Conditions
2. Noise

where

1. The states

1. The correlation

which implies

1. The mean and covariance

* The mean
* The covariance
* The **transition** matrix – notation abuse
* We will discuss in the next Tuesday. However in discrete linear system(or Markov process) there is a definition of the transition matrix.
  1. Non-linear Stochastic Difference Equations 🡪 skip